

EASY TURBULENCE¹

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It seems a safe bet that the understanding of developed turbulence, a long standing challenge for theoretical and mathematical physics, will enter into the third millennium as an unsolved problem. This is an introductory course to the subject. We discuss

in **Lecture 1**: the Navier Stokes equations, existence of solutions, statistical description, energy balance and cascade picture;

in **Lecture 2**: the Kolmogorov theory of three-dimensional turbulence versus intermittency, the Kraichnan-Batchelor theory of two-dimensional turbulence;

in **Lecture 3**: the Richardson dispersion law and the breakdown of the Lagrangian flow;

in **Lecture 4**: direct and inverse cascades and intermittency in the Kraichnan model of passive advection.

LECTURE 1

Theoretical physics pursues two goals. On one side, it searches for fundamental laws of nature. On the other side, it studies the theoretical and phenomenological consequences of the laws already found. Hydrodynamics represents the second case. Its fundamental laws, in the form of the Navier-Stokes (NS) equation or its variations, were formulated in the first half of the nineteenth century by Navier (1823) and Stokes (1843) as a modification of the even older Euler equation (1755). The NS equation describes the temporal evolution of a **velocity** field $\mathbf{v}(t, \mathbf{x})$ in gasses or liquids. It takes the form

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu \nabla^2 \mathbf{v} = \frac{1}{\rho} (\mathbf{f} - \nabla p), \quad (1)$$

where ν is the viscosity of the fluid ($\cong 1.5 \times 10^{-5} \frac{\text{m}^2}{\text{sec}}$ for air, $\cong 10^{-6} \frac{\text{m}^2}{\text{sec}}$ for water), ρ is the fluid density, \mathbf{f} is the external (intensive) force and p is the pressure. In most physical applications, the dimensionality of the space is 3 or 2, but the equations make sense in a general dimension d .

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The Euler equation without the $\nu \nabla^2 \mathbf{v}$ term is really the $\mathbf{F} = m\mathbf{a}$ (or rather $\mathbf{a} = \frac{1}{m}\mathbf{F}$) relation for the volume element of the fluid. The $\nu \nabla^2 \mathbf{v}$ term in the NS equation represents the friction forces. The relation (1) has to be supplemented with the continuity equation $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$, and an equation of state relating ρ and p . In most applications, one may assume that the fluid is **incompressible**, i.e. that ρ is constant and that \mathbf{v} is divergence free²:

$$\nabla \cdot \mathbf{v} = 0. \quad (2)$$

It follows then by taking the divergence of both sides of Eq. (1) that $\nabla^2 p = -\nabla \cdot (\mathbf{v} \cdot \nabla) \mathbf{v}$ so that pressure may be calculated for given velocity field. It may be also eliminated by applying to the NS equation the transverse projector which leaves the divergenceless \mathbf{v} unchanged. The incompressible Euler equation has a nice infinite-dimensional geometric interpretation: it describes the geodesic flow on the group of volume preserving diffeomorphisms³. In one space dimension and without the incompressibility constraint and the pressure term, Eq. (1) becomes the Burgers equation, one of the simplest non-linear equations.

The incompressible Euler and NS equations are examples of nonlinear partial differential evolution equations. After a century and a half of studies, they still pose major open problems as far as the control of their solutions is concerned. The most interesting questions touch on the short-distance (ultra-violet) behavior. Suppose that we start from smooth initial data and the force \mathbf{f} is smooth. For simplicity, let us assume compact support of both (we may also consider the compact space or the case with boundary conditions). It is known that the smooth solutions of the so posed initial value problem are unique and exist for short time. Do they exist for all times? The answer is positive in 2 dimensions for both Euler and NS equations but in 3 dimensions the answer is not known. It is usually expected to be positive in the NS case. The opinions about the Euler case (no blowup versus finite-time blowup for special smooth initial conditions) are more divided and fluctuate in time.

In an important 1933 paper on the NS equation, Leray has introduced the notion of weak solutions of the equation. A vector field $\mathbf{v}(t, x)$ locally in L^2 is a weak solution if it satisfies the equations in the distributional sense, i.e. if for any smooth vector field \mathbf{u} without divergence and any smooth function φ , both with compact supports,

$$\int \left[\left(\partial_t u^i + \nu \Delta u^i + (\partial_j u^i) v^j \right) v^i + u^i f^i \right] = 0 \quad \text{and} \quad \int (\partial_i \varphi) v^i = 0.$$

Leray showed by compactness arguments existence of global weak solutions of the 3-dimensional NS equations with additional properties (e.g. with space derivatives locally square integrable). The weak solutions are not unique (there are weak solutions of the 2-dimensional Euler equation with compact support).

The NS equation is invariant under rescalings. Let

$$\begin{aligned} \tilde{\mathbf{v}}(t, \mathbf{x}) &= \tau s^{-1} \mathbf{v}(\tau t, s\mathbf{x}), \\ \tilde{\mathbf{f}}(t, \mathbf{x}) &= \tau^2 s^{-1} \mathbf{f}(\tau t, s\mathbf{x}), \\ \tilde{p}(t, \mathbf{x}) &= \tau^2 s^{-2} p(\tau t, s\mathbf{x}), \\ \tilde{\nu} &= \tau s^{-2} \nu. \end{aligned}$$

If \mathbf{v} and p solve the NS equation with viscosity ν and force \mathbf{f} then $\tilde{\mathbf{v}}$ and \tilde{p} give a solution for viscosity $\tilde{\nu}$ and force $\tilde{\mathbf{f}}$. It is convenient to introduce the dimensionless version of the (inverse)

²we shall absorb in this case the constant $\frac{1}{\rho}$ into p and \mathbf{f}

³recall that the Euler top is related to the geodesic flow on the group $SO(3)$

viscosity, the Reynolds number

$$Re = \frac{L \delta_L v}{\nu},$$

where $\delta_L v$ is a characteristic size of velocity differences over scale L of the order of the size of the system. Note the scale-dependent nature of the concept. For the flow in a pipe of radius L , we may take $\delta_L v$ as velocity in the middle of the pipe (the velocity vanishes on the wall of the pipe). The following are the basic phenomenological observations about hydrodynamics. If $Re \ll 1$, one encounters regular ("**laminar**") flows. For Re between ~ 1 and $\sim 10^2$, one observes complicated phenomena depending on the precise situation. For $Re \gg 10^2$, very irregular ("**turbulent**") flows set in. They show for high Re ("**developed turbulence**") a certain degree of similarity for different circumstances.

Somewhat simplifying, one could say that for the laminar flows the non-linear term $(\mathbf{v} \cdot \nabla) \mathbf{v}$ of the NS equation plays a smaller role. This is a relatively well understood regime, also rigorously. Following Gallavotti's article cited at the end of the Lecture, define the running Reynolds number

$$Re_r = \frac{r^2}{\nu} \left(\frac{1}{|B_r|} \int_{B_r} |\nabla \mathbf{v}|^2 \right)^{1/2}$$

where $B_r = \{ (s, \mathbf{y}) \mid |s - t| < \frac{r^2}{\nu}, |\mathbf{y} - \mathbf{x}| < r \}$ is a neighborhood of the space-time point (t, \mathbf{x}) . Note that we may rewrite

$$Re_r = \frac{r \delta_r v}{\nu}$$

where $\delta_r v$, the mean velocity difference on scale r , is calculated by multiplying the mean square gradient of v over B_r by r . The best regularity result about the weak solutions of the NS equation is due to Caffarelli-Kohn-Nirenberg and says that there exists $\epsilon > 0$ such that if $Re_r \leq \epsilon$ then the solution is smooth in the $B_{\epsilon r}$ neighborhood of (t, \mathbf{x}) . This implies that, for a weak solution, the Hausdorff dimension of the set of singularities is ≤ 1 . Note the spirit of the result in line with the phenomenological characterization of laminar flows.

One expects that in the regime of intermediate Re between ~ 1 and $\sim 10^2$ only a finite number of modes of fluid play an important role. These modes may be effectively described by ordinary differential equations to which the theory of dynamical systems (bifurcations, strange attractors, chaology) may be applied. Indeed, the dynamical system theory has been used with much success to describe specific situations, as the flow between rotating cylinders, for example. It is not clear, however, if the dynamical system ideas may be useful to describe the fully developed turbulence.

The importance of the NS equations is far from being limited to the mathematical questions. It extends to meteorology, aeronautics and maritime engineering, to mention only three of the most important domains of practical applications. The regime of large Re , where the non-linear term of the NS equation becomes very important dominates in practical situations (in medium size river, $Re \sim 10^7$). One has to admit that the interests of engineers and theoretical physicists are somewhat different. The first ones are interested mainly in flows around obstacles (e.g. an airplane wing) whereas the second ones show a tendency to concentrate on flows far from boundaries where the simplifying assumptions of homogeneity and isotropy seem in

place. Nevertheless, a good understanding of such idealized flows would certainly have practical consequences.

For high Reynolds numbers, it is reasonable to attempt a statistical description of complicated turbulent flows. In the theoretical approach, the statistics may be generated by considering random initial data or/and random forcing. Since some degree of universality is observed in this situation independently on the way the flow is excited, one often assumes that the force \mathbf{f} is a random Gaussian field with mean zero and the covariance

$$\langle f^i(t, \mathbf{x}) f^j(s, \mathbf{y}) \rangle = \delta(t - s) \chi^{ij}(\frac{\mathbf{x} - \mathbf{y}}{L}) \quad (3)$$

with $\partial_i \chi^{ij} = 0$ and the **injection scale** L regulating the decay of χ , i.e. the scale on which the force acts. $\mathbf{v}(t, \mathbf{x})$ becomes then a random field and the NS equation takes, schematically, the form

$$\partial_t \Phi = -F(\Phi) + \eta. \quad (4)$$

Many other dynamical problems in physics may be put in such a form with $F(\Phi)$ being a nonlinear functional of local densities $\Phi(t, \mathbf{x})$ of physical quantities and η a random noise. The case of the NS equation should be contrasted with another example of Eq. (4), provided by the Langevin equation describing the approach to equilibrium in systems of statistical mechanics or field theory. In the latter case, the nonlinearity is of the gradient type:

$$F(\Phi) = \frac{\delta S(\Phi)}{\delta \Phi}$$

with $S(\Phi)$ a local functional, e.g. $S(\Phi) = \frac{1}{2} \int (\nabla \Phi)^2 + \frac{1}{2} m^2 \int \Phi^2 + \lambda \int \Phi^4$ in the Φ^4 field theory. The noise is taken Gaussian:

$$\langle \eta(t, \mathbf{x}) \eta(s, \mathbf{y}) \rangle = \delta(t - s) L^{-d} \chi(\frac{\mathbf{x} - \mathbf{y}}{L}). \quad (5)$$

The covariance $L^{-d} \chi(\mathbf{x}/L)$ regulates the theory on short distances $\lesssim L$ and is close to the delta-function $\delta(\mathbf{x})$ for small L . On the contrary, in the case of the NS equations we are mostly interested in forces acting on large distances $\sim L$ (e.g. the convective forces in the atmosphere) so that the force covariance $\chi^{ij}(\mathbf{x}/L)$ becomes close to a constant in the position space, i.e. to a multiple of the delta-function in the wavenumber space. Such regime in field theory would correspond to distances shorter than the ultraviolet cutoff, with the behavior strongly dependent on the detailed form of the cutoff. Another difference is that in Eq. (1) the non-linear term, dominant for high Reynolds numbers, is not of the gradient type. Finally, the presence of pressure renders it also nonlocal, which is another complication. All these differences make the case of the NS equation quite different from that of the Langevin equation describing, in the stationary regime, equilibrium states. They make the NS problem, strongly coupled for high Re , resistant to the methods employed successfully in the study of equilibrium states like perturbative approaches or the renormalization group.

The main characteristic of the stationary regimes of the randomly forced NS equations is the presence of non-vanishing fluxes of conserved quantities, forbidden in equilibrium states. By integrating the scalar product of the incompressible NS equation with \mathbf{v} over the space and assuming that the flow velocity vanishes sufficiently fast at large distances (or at the boundary), one deduces the energy balance:

$$\partial_t \frac{1}{2} \int \mathbf{v}^2 = -\nu \int (\nabla \mathbf{v})^2 + \int \mathbf{f} \cdot \mathbf{v}. \quad (6)$$

The equation states that the rate of change of fluid energy is equal to the energy injection rate $\int \mathbf{f} \cdot \mathbf{v}$ (work of the external forces per unit time) minus the energy dissipation per unit time $\nu \int (\nabla \mathbf{v})^2$ due to the viscous friction. In a stationary state, the mean overall energy of the fluid is constant in time so that the energy balance equation (6) implies that

$$\int \langle \nu (\nabla \mathbf{v})^2 \rangle = \int \langle \mathbf{v} \cdot \mathbf{f} \rangle,$$

where $\langle - \rangle$ denotes the ensemble average, or, that in the homogeneous state,

$$\bar{\epsilon} \equiv \langle \nu (\nabla \mathbf{v})^2 \rangle = \langle \mathbf{v} \cdot \mathbf{f} \rangle \equiv \bar{\varphi} \quad (7)$$

where $\bar{\epsilon}$ denotes the mean dissipation rate and $\bar{\varphi}$ the mean injection rate of energy, both with the dimension $\frac{\text{length}^2}{\text{time}^3}$. In the situation where the energy injection is a large distance process (e.g. in the atmospheric turbulence or shear flows) one expects that for high Re a **scale separation** occurs, with the energy dissipation taking place on much smaller distances. Pictorially, energy is transmitted to the fluid by the excitation of large eddies on scale L which subsequently break to smaller scale eddies and so on. This way the injected energy is passed to shorter and shorter scales without substantial loss, until the viscous scale η is reached where the friction dissipates energy. Such an **energy cascade**, described first by Richardson in 1922, is characterized by the integral scale L , the viscous scale η and the energy dissipation rate $\bar{\epsilon}$. The scale ratio L/η should grow with the Reynolds number. The interval of distance scales r satisfying $L \gg r \gg \eta$ is called the **inertial range**.

The cascade picture may be formulated in more quantitative terms by introducing the quantities

$$\begin{aligned} \bar{\epsilon}_{\leq K} &= \nu \int_{|\mathbf{k}| \leq K} \left(\int e^{-i\mathbf{k} \cdot \mathbf{x}} \langle \nabla \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{0}) \rangle d\mathbf{x} \right) d\mathbf{k}, \\ \bar{\varphi}_{\leq K} &= \int_{|\mathbf{k}| \leq K} \left(\int e^{-i\mathbf{k} \cdot \mathbf{x}} \langle \mathbf{v}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{0}) \rangle d\mathbf{x} \right) d\mathbf{k} \end{aligned}$$

interpreted as the mean dissipation and mean injection rate in wavenumbers \mathbf{k} with $|\mathbf{k}| \leq K$ ($d\mathbf{k} \equiv \frac{d\mathbf{k}}{(2\pi)^d}$). The injection of energy limited to distances $\gtrsim L$ means that, as a function of K , $\bar{\varphi}_{\leq K}$ is close to $\bar{\epsilon}$ everywhere except for $K \lesssim \frac{1}{L}$ where it falls to zero with $K \rightarrow 0$. E.g., for the time-decorrelated force with an appropriate interpretation of Eq. (1) as a stochastic differential equation, the relation $\langle \mathbf{v}(t, \mathbf{x}) \mathbf{f}(t, \mathbf{0}) \rangle = \frac{1}{2} \text{tr} \chi(\frac{\mathbf{x}}{L})$ holds so that $\bar{\varphi}_{\leq K} = \frac{1}{2} \int_{|\mathbf{k}| \leq K} \left(\int e^{-i\mathbf{k} \cdot \mathbf{x}} \text{tr} \chi(\frac{\mathbf{x}}{L}) d\mathbf{x} \right) d\mathbf{k}$ and such behavior of $\bar{\varphi}_{\leq K}$ follows since χ is close to a delta function in the wavenumber space. The cascade picture should imply that the mean dissipation rate $\bar{\epsilon}_{\leq K}$ is negligible for $K \ll \frac{1}{\eta}$ and then grows to $\bar{\epsilon}$. The difference

$$\bar{\varphi}_{\leq K} - \bar{\epsilon}_{\leq K} \equiv \bar{\pi}_K$$

has the interpretation of the energy flux out of the wavenumbers \mathbf{k} with $|\mathbf{k}| \leq K$. This flux should be approximately constant and equal to $\bar{\epsilon}$ in the inertial range $\frac{1}{L} \ll K \ll \frac{1}{\eta}$, see Fig. 1.

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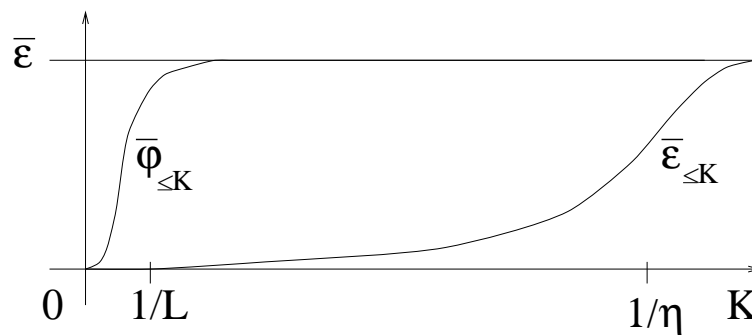


Fig. 1

LECTURE 2

In 1941, A. N. Kolmogorov has proposed a scaling theory of the developed turbulence which has deeply marked the ensuing turbulence research and rests a reference point of most of the modern research on the subject. It is based on the exact statistical relations for the turbulent velocities obtained with general assumptions and going back to the 1938 work of von Kármán and Horwarth. The NS equation implies that

$$\mathbf{v}(t + \Delta t) = \mathbf{v}(t) + [-(\mathbf{v} \cdot \nabla \mathbf{v})\mathbf{v} + \nu \nabla^2 \mathbf{v} - \nabla p]_t \Delta t + \int_t^{t+\Delta t} \mathbf{f}(s) ds + \mathcal{O}((\Delta t)^2)$$

(the term involving the white noise in time \mathbf{f} is $\mathcal{O}(t^{1/2})$). Consequently, we obtain for the statistical expectation of the scalar product of two velocities at the same time but general points:

$$\begin{aligned} \langle \mathbf{v}(t + \Delta t, \mathbf{x}) \cdot \mathbf{v}(t + \Delta t, \mathbf{y}) \rangle &= \langle \mathbf{v}(t, \mathbf{x}) \cdot \mathbf{v}(t, \mathbf{y}) \rangle \\ &+ \left[- \langle (\mathbf{v} \cdot \nabla) \mathbf{v}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{y}) \rangle - \langle \mathbf{v}(\mathbf{x}) \cdot (\mathbf{v} \cdot \nabla) \mathbf{v}(\mathbf{y}) \rangle \right. \\ &\left. + \nu \langle \nabla^2 \mathbf{v}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{y}) \rangle + \nu \langle \mathbf{v}(\mathbf{x}) \cdot \nabla^2 \mathbf{v}(\mathbf{y}) \rangle + \text{tr} \chi\left(\frac{\mathbf{x}-\mathbf{y}}{L}\right) \right]_t \Delta t + \mathcal{O}((\Delta t)^2). \end{aligned}$$

We have dropped the pressure terms assuming the homogeneity (i.e. translation invariance) of the statistical state. The second line terms may be rewritten as

$$\frac{1}{2} \nabla_{\mathbf{x}} \cdot \langle (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))(\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))^2 \rangle$$

and the next two ones as

$$- 2\nu \langle \nabla \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{y}) \rangle.$$

Equating the $\mathcal{O}(\Delta t)$ terms, we obtain

$$\begin{aligned} \partial_t \langle \mathbf{v}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{y}) \rangle &= \frac{1}{2} \nabla_{\mathbf{x}} \cdot \langle (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))(\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))^2 \rangle \\ &- 2\nu \langle \nabla \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{y}) \rangle + \text{tr} \chi\left(\frac{\mathbf{x}-\mathbf{y}}{L}\right), \end{aligned} \quad (8)$$

which is the basic relation between the 2-point and the 3-point correlation functions of velocity.

In three (and more) dimensions, we expect the stabilization of the correlation functions for long times. In the stationary regime, the time derivatives of the equal-time functions vanish and we infer that

$$- \frac{1}{4} \nabla_{\mathbf{x}} \cdot \langle (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))(\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))^2 \rangle + \nu \langle \nabla \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{y}) \rangle = \frac{1}{2} \text{tr} \chi\left(\frac{\mathbf{x}-\mathbf{y}}{L}\right). \quad (9)$$

Taking first the limit $\mathbf{y} \rightarrow \mathbf{x}$ for positive ν and assuming that the presence of the latter smoothes out the behavior of $\langle (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))(\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))^2 \rangle$ so that the first term on the left hand side vanishes in the limit, we obtain

$$\bar{\epsilon} = \frac{1}{2} \text{tr} \chi(\mathbf{0})$$

which is nothing else but the energy balance equation (7) for the case of the time decorrelated force injecting energy at the mean rate $\bar{\varphi} = \frac{1}{2} \text{tr} \chi(\mathbf{0})$.

To deduce further implications of Eq. (9), we take its invicid limit $\nu \rightarrow 0$ keeping the points separate. This gives

$$-\frac{1}{4} \nabla_{\mathbf{x}} \cdot \langle (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))(\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))^2 \rangle|_{\nu=0} = \frac{1}{2} \text{tr} \chi\left(\frac{\mathbf{x}-\mathbf{y}}{L}\right). \quad (10)$$

The assumption that the force acts only on distances $\gtrsim L$ means that, for $|\mathbf{x}-\mathbf{y}| \ll L$, $\chi(\frac{\mathbf{x}-\mathbf{y}}{L}) = \chi(\mathbf{0})$, approximately, so that Eq. (10) implies that in the inertial range,

$$-\frac{1}{4} \nabla_{\mathbf{x}} \cdot \langle (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))^2 (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})) \rangle = \bar{\epsilon}.$$

Assuming the isotropy (i.e. the rotation invariance), this implies the relation

$$\langle (v^i(\mathbf{x}) - v^i(\mathbf{y}))(v^j(\mathbf{x}) - v^j(\mathbf{y}))(v^k(\mathbf{x}) - v^k(\mathbf{y})) \rangle = -\frac{4\bar{\epsilon}}{d(d+2)} (\delta^{ij} r^k + \delta^{ik} r^j + \delta^{jk} r^i),$$

where $\mathbf{r} = \mathbf{x} - \mathbf{y}$. In other words, the 3-point function of equal-time velocity difference is in the inertial range linear in the point separation. In particular, for the so called longitudinal 3-point structure function, we obtain

$$S_3^{\parallel}(r) \equiv \langle (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})) \cdot \frac{\mathbf{r}}{r} \rangle^3 = -\frac{12}{d(d+2)} \bar{\epsilon} r \quad (11)$$

known, for $d = 3$, as the **Kolmogorov four-fifths law**.

One may deduce a stronger version of the above relation which takes the form of the operator product expansion for the $\nu \rightarrow 0$ limit of the dissipation operator $\epsilon = \nu(\nabla \mathbf{v})^2$:

$$\epsilon(\mathbf{x}) = -\frac{1}{4} \lim_{\mathbf{y} \rightarrow \mathbf{x}} \nabla_{\mathbf{x}} \cdot [(\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))(\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))^2] \Big|_{\nu=0} \quad (12)$$

which should hold inside expectations in the $\nu \rightarrow 0$ limit. As noticed recently by Duchon and Robert, the relation (12) holds for all weak solutions of the Euler equation which are limits of strong solutions of the NS equation. Relation (12) is often called a **dissipative anomaly**: the dissipation ϵ whose definition involves a factor of ν does not vanish when $\nu \rightarrow 0$.

In his 1941 paper, Kolmogorov went one step further by postulating the universal character of the turbulent cascade in the inertial range with the equal-time correlators of velocity differences over inertial range distances given by universal functions of the latter and of the dissipation rate $\bar{\epsilon}$. In particular this implies that the velocity structure functions $S_n^{\parallel}(r) = \langle (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})) \cdot \frac{\mathbf{r}}{r} \rangle^n$ are determined by dimensional reasons up to universal constants:

$$S_n^{\parallel}(r) = C_n \bar{\epsilon}^{n/3} r^{n/3}. \quad (13)$$

Indeed, the right hand side is the only function of $\bar{\epsilon}$ and r with the dimension $(\frac{\text{length}}{\text{time}})^n$. The physical content of the Kolmogorov theory is that the typical velocity v_r of size r eddies behaves as $\bar{\epsilon}^{1/3} r^{1/3}$. During the **eddy turnover time** $\tau_r = \frac{r}{v_r} \propto \bar{\epsilon}^{-1/3} r^{2/3}$, these eddies transfer their energy with the density $\frac{1}{2} v_r^2 \propto \bar{\epsilon}^{2/3} r^{2/3}$ to the shorter scale resulting in the constant energy flux $\propto \bar{\epsilon}$. For the scale-dependent Reynolds number one obtains then $Re_r \propto \frac{\bar{\epsilon}^{1/3} r^{4/3}}{\nu}$. In particular, $Re = Re_L \propto \frac{\bar{\epsilon}^{1/3} L^{4/3}}{\nu}$ and $1 = Re_{\eta} \propto \frac{\bar{\epsilon}^{1/3} \eta^{4/3}}{\nu}$. Hence $\frac{\eta}{L} \propto Re^{-3/4}$ and it decreases with Re .

As we have seen, the $n = 3$ relation (13) coincides with the Kolmogorov four-fifths law. The structure functions are measured, more or less directly, in atmospheric or ocean flows, in

water jets, in aerodynamic tunnels or in subtle experiments with helium gas in between rotating cylinders or plates. They are also accessible in numerical simulations. One extracts then the scaling exponents assuming the behavior

$$S_n^{\parallel}(x) \propto r^{\zeta_n}.$$

ζ_3 agrees well with the theoretical prediction $\zeta_3 = 1$. Here are some other exponents obtained from wind tunnel data

$$\zeta_2 = .70(.67), \quad \zeta_4 = 1.28(1.33), \quad \zeta_5 = 1.53(1.67), \quad \zeta_6 = 1.77(2), \quad \zeta_7 = 2.01(2.33)$$

with the Kolmogorov values in the parenthesis for comparison. The discrepancy is quite pronounced (its direction for the even functions is determined by the Hölder inequality implying that ζ_n is a concave function of n). One of the main open problems in the theory of fully developed turbulence is to explain, starting from the first principles (i.e. from the NS equation), the breakdown of the Kolmogorov theory leading to the anomalous structure-function exponents which indicate that the distribution of $\mathbf{v}(t, \mathbf{x})$ in the inertial range is rather different from Gaussian. The discrepancy may be measured by the **skewness** $S_3/S_2^{3/2}$ or the **flatness** S_4/S_2^2 which grow with diminishing distance instead of being equal to their Gaussian values 0 and 3. Hence the domination of the short scales by the large deviations of the velocity differences indicating an enhanced short-distance activity: the phenomenon called **intermittency**. The intuitive explanation of intermittency which was advanced is that only a part of the fluid modes (temporal or/and spatial) participates in the turbulent cascade, with the proportion of active modes decreasing with diminishing scale. This forces the active short-distance modes to transfer more energy and, consequently, to be more excited. This picture led to multiple (multi)fractal models of the cascade reviewed in the book by Frisch cited at the end of this Lecture. Such models, although interesting phenomenologically, are not based on the NS equation and allow to obtain essentially arbitrary spectra of exponents. They do not really explain the mechanism of the breakdown of the normal scaling in realistic flows.

For the **energy spectrum**

$$\bar{e}_K \equiv \frac{1}{2} \frac{d}{dK} \int_{|\mathbf{k}| \leq K} \left(\int \langle \mathbf{v}(\mathbf{x}) \cdot \mathbf{v}(0) \rangle e^{-i\mathbf{k} \cdot \mathbf{x}} d\mathbf{x} \right) d\mathbf{k},$$

the Kolmogorov theory predicts

$$\bar{e}_K \propto \bar{\epsilon}^{2/3} K^{-5/3} \quad (14)$$

for $\frac{1}{L} \ll K \ll \frac{1}{\eta}$ just by the dimensional count. The experimental data seem to confirm this behavior (with the possibility of a slight discrepancy consistent with the value of ζ_2 cited above).

For the mean dissipation rate, we obtain: $\bar{e}_{\leq K} = 2\nu \int_0^K K'^2 \bar{e}_{K'} dK' \propto \nu \bar{\epsilon}^{2/3} K^{4/3}$ confirming that $\bar{e}_{\leq K}/\bar{\epsilon} = Re_{K^{-1}}^{-1} \ll Re_{\eta}^{-1} = 1$ in the inertial range. Deep in the dissipative regime $K \gg \frac{1}{\eta}$, \bar{e}_K falls off much faster than in the inertial interval.

An important object in the turbulence theory is the **vorticity** field that measures the strength and the orientation of eddies. In three dimensions it is a (pseudo-)vector field $\omega = \nabla \times \mathbf{v}$ and it satisfies, for the incompressible flow, the equation

$$\partial_t \omega + (\mathbf{v} \cdot \nabla) \omega - (\omega \cdot \nabla) \mathbf{v} - \nu \nabla^2 \omega = \nabla \times \mathbf{f}. \quad (15)$$

Note that $(\mathbf{v} \cdot \nabla)\omega - (\omega \cdot \nabla)\mathbf{v}$ is the commutator of two vector fields, so that, at vanishing ν and \mathbf{f} , the equation (15) implies that the vorticity is transported by velocity field as a vector. This involves stretching of ω by the velocity **strain** $\frac{1}{2}(\partial_i v^j + \partial_j v^i)$, an important mechanism for the energy transfer between scales. In two dimensions, however, the vorticity reduces to a (pseudo-)scalar field $\omega = \epsilon^{ij} \partial_i v^j$ whose evolution is governed by the equation

$$\partial_t \omega + (\mathbf{v} \cdot \nabla)\omega - \nu \nabla^2 \omega = \epsilon^{ij} \partial_i f^j. \quad (16)$$

Here, at vanishing ν and \mathbf{f} , the vorticity is simply transported by the velocity field along the **Lagrangian trajectories** $\mathbf{x}(t)$ of the (imaginary) fluid particles s.t. $\frac{d\mathbf{x}}{dt} = \mathbf{v}(t, \mathbf{x})$. In particular, the two-dimensional flow, unlike the three-dimensional one, conserves, besides energy, the **enstrophy** $\Psi = \frac{1}{2} \int \omega^2$ (as well as the integrals of higher powers of vorticity).

In the seminal 1968 paper, R. H. Kraichnan has realized that the conservation of enstrophy implies a very different cascade picture in two dimensions, as compared to the three-dimensional one, see also the paper of Batchelor of 1969. First of all, the **enstrophy spectrum** is related to the energy spectrum:

$$\bar{\psi}_K \equiv \frac{1}{2} \frac{d}{dK} \int_{|\mathbf{k}| \leq K} \left(\int \langle \omega(\mathbf{x}) \omega(\mathbf{0}) \rangle e^{-i\mathbf{k} \cdot \mathbf{x}} d\mathbf{x} \right) d\mathbf{k} = K^2 \bar{e}_K.$$

Similarly, the mean enstrophy flux out of wavenumbers \mathbf{k} with $|\mathbf{k}| \leq K$, if local in the momentum space, would have to be equal K^2 times the energy flux $\bar{\pi}_K$ so that the constancy of both fluxes is impossible. Kraichnan reasoned that, in such a situation, it will be the enstrophy flux which is constant on scales smaller than the injection scale so that the small scales will support a **direct enstrophy cascade** towards smaller and smaller distances, with all the enstrophy dissipation taking place on the shortest scales. The energy flux towards small scales will then be damped and, as a result, the energy will be, instead, transferred to scales longer than the injection scale L in an **inverse energy cascade** process. If not impaired by boundaries or large scale friction, this process would lead to the pumping of energy into the constant mode at the rate equal to $\frac{1}{2} \text{tr} \chi(\mathbf{0})$.

If we assume that $\langle \mathbf{v}(t, \mathbf{x}) \cdot \mathbf{v}(t, \mathbf{y}) \rangle - t \text{tr} \chi(\mathbf{0})$ stabilizes at long times, then Eq. (8) will lead to the relation

$$\frac{1}{4} \nabla_{\mathbf{x}} \cdot \langle (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})) (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))^2 \rangle - \nu \langle \nabla \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{y}) \rangle = \frac{1}{2} \text{tr} [\chi(\mathbf{0}) - \chi(\frac{\mathbf{x}-\mathbf{y}}{L})]. \quad (17)$$

In particular, in the limit $\nu \rightarrow 0$ and for $|\mathbf{x} - \mathbf{y}| \gg L$,

$$\frac{1}{4} \nabla_{\mathbf{x}} \cdot \langle (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y})) (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))^2 \rangle = \frac{1}{2} \text{tr} \chi(\mathbf{0})$$

implying the relation similar to the two-dimensional version of Eq. (11) but with the inverted sign. In particular, we obtain

$$S_3^{\parallel}(r) = \frac{3}{2} \bar{\epsilon} r \quad \text{for } r \gg L,$$

a **three-halves law**.

The inverse cascade of the two-dimensional turbulence has been recently under an intensive study. Both experimental and numerical data confirm the above prediction and indicate that

in this regime all structure functions, although non-Gaussian, scale with the Kolmogorov exponents $\zeta_n = \frac{n}{3}$ indicating that the **inverse cascade is not intermittent**. This implies for the energy spectrum, the behavior (14) for $K \ll \frac{1}{L}$.

For $|\mathbf{x} - \mathbf{y}| \ll L$, Eq. (17) reduces in the $\nu \rightarrow 0$ limit to the relation

$$\frac{1}{4} \nabla_{\mathbf{x}} \cdot \langle (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))(\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{y}))^2 \rangle = \frac{1}{2} \text{tr} [\chi(\mathbf{0}) - \chi(\frac{\mathbf{x}-\mathbf{y}}{L})] = \mathcal{O}(r^2) \quad (18)$$

implying the scaling

$$S_3^{\parallel}(r) \sim r^3 \quad \text{for } r \ll L,$$

i.e. in the direct cascade regime. It is possible to infer in two dimensions another exact relation for the 3-point functions by proceeding from Eq. (16) the same way that we did before. One obtains

$$\begin{aligned} \frac{1}{2} \partial_t \langle \omega(t, \mathbf{x}) \omega(t, \mathbf{y}) \rangle &= \frac{1}{4} \nabla_{\mathbf{x}} \cdot \langle (\mathbf{v}(t, \mathbf{x}) - \mathbf{v}(t, \mathbf{y})) (\omega(t, \mathbf{x}) - \omega(t, \mathbf{y}))^2 \rangle \\ &\quad - \nu \langle \nabla \omega(t, \mathbf{x}) \cdot \nabla \omega(t, \mathbf{y}) \rangle - \frac{1}{2} \nabla_{\mathbf{x}}^2 \text{tr} \chi(\frac{\mathbf{x}-\mathbf{y}}{L}). \end{aligned} \quad (19)$$

Our previous assumption about the stabilization of the velocity 2-point function modulo a growing constant implies that the 2-point function of the vorticity reaches a stationary regime. We deduce from this, as before for the three-dimensional velocities, that the mean dissipation rate of vorticity

$$\bar{\epsilon}_{\omega} \equiv \nu \langle (\nabla \omega)^2 \rangle = -\frac{1}{2} \nabla_{\mathbf{x}}^2 \text{tr} \chi(\mathbf{0})$$

is ν -independent and that, in the limit $\nu \rightarrow 0$,

$$-\frac{1}{4} \nabla_{\mathbf{x}} \cdot \langle (\mathbf{v}(t, \mathbf{x}) - \mathbf{v}(t, \mathbf{y})) (\omega(t, \mathbf{x}) - \omega(t, \mathbf{y}))^2 \rangle = -\frac{1}{2} \nabla_{\mathbf{x}}^2 \text{tr} \chi(\frac{\mathbf{x}-\mathbf{y}}{L}). \quad (20)$$

As noticed in a recent paper of Bernard, see the bibliography, the relations (18) and (20), both consistent with the $\mathcal{O}(r^3)$ behavior of the 3-point functions of velocity differences in the direct cascade regime, allow to find exactly its asymptotic form. If we assume next that the n -point functions of the velocity differences scale accordingly, i.e. with the power n , then, for $n = 2$, we infer the Kraichnan-Batchelor energy spectrum in the direct cascade

$$\bar{e}_K \sim K^{-3} \quad \text{for } K \gg \frac{1}{L}, \quad (21)$$

confirmed by experimental observations. Intermittency, if existent in two-dimensional turbulence, seems much smaller than in three dimensions, especially in the inverse cascade. There are, however, theoretical predictions of logarithmic corrections to the power-law scaling in the direct cascade, not yet accessible to experimental or numerical verification.

Summarizing: In three dimensions, in the inertial range of the direct (short distance) energy cascade, the 3-point velocity structure function scales linearly in the distance. The energy spectrum is close to $\propto K^{-5/3}$, with the anomalous scaling of higher-point structure functions signaling intermittency. In two dimensions, in the direct (short distance) enstrophy cascade, the 3-point structure function scales as the 3rd power of the distance and the energy spectrum is close to $\propto K^{-3}$. In the (long distance) inverse energy cascade, one observes the Kolmogorov scaling, similarly as in the three dimensional direct cascade, but with reduced

intermittency. Assuming the above spectra, one infers that both in two and in three dimensions, the mean enstrophy density given by $\int K^2 \bar{e}_K dK$ diverges in the ultraviolet in the invicid limit $\nu \rightarrow 0$. Accordingly, one should expect that the stationary fully turbulent state is carried by weak solutions of the Euler equation with divergent enstrophy. Hence the importance of studying such solutions which, in three dimensions, dissipate energy on short distances by a mechanisms which is the topic of next Lecture.

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LECTURE 3

One of the first quantitative laws of fully developed turbulence has been formulated by L. F. Richardson in 1926. Basing on observations, e.g. of the movements of balloon probes released in the atmosphere, Richardson noted that the mean rate of growth of the square of the separation between two such probes is proportional to the four-thirds power of the distance, instead of being distance-independent, as in the Brownian diffusion. In other words, the Richardson dispersion law states that

$$\frac{d\rho^2}{dt} \propto \rho^{4/3}, \quad (22)$$

where $\rho(t)$ is the time t distance between two Lagrangian trajectories satisfying the ordinary differential equation (ODE)

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(t, \mathbf{x}). \quad (23)$$

Of course, the Richardson law is compatible with the Kolmogorov theory that implies that the velocity differences $|\mathbf{v}(t, \mathbf{x}) - \mathbf{v}(t, \mathbf{y})|$ scale as $\rho^{1/3}$ for $\rho = |\mathbf{x} - \mathbf{y}|$ in the inertial range, but it deserves a more detailed discussion.

Let us note that Eq. (22) is solved by

$$\rho^{2/3} = \rho_0^{2/3} + C t$$

which, in the limit when the initial (i.e. time zero) separation goes to zero, reduces to the relation

$$\rho^2 \propto t^3. \quad (24)$$

Of course, some care is needed since the law (22) was observed for separations in the inertial range, so for $\rho \gg \eta$, where η is the viscous scale. In the limit $Re \rightarrow \infty$, however, η tends to zero and the growth (24) should hold for arbitrarily short separations. In the turbulent atmosphere, for example, the viscous scale η is of the order of millimeter so that neglecting it when we look at scales of meters or even kilometers is a reasonable approximation. We infer this way that at $Re = \infty$,

infinitesimally close Lagrangian trajectories separate in finite time.

Of course, ρ is the average separation, with the mean taken over different velocity fields of the statistical turbulent ensemble. One expects, however, that the trajectory separation is a self-averaging quantity so that this type of behavior holds already in a fixed typical velocity realization if we average over initial positions and times of release of two very close trajectories. What is this strange behavior of the trajectories? We are not used to such behaviors of the solutions of the ODE's. For example, in the integrable dynamical systems the distance between two trajectories

$$\rho \simeq \mathcal{O}(1).$$

In the dissipative systems,

$$\rho \simeq \mathcal{O}(e^{-\lambda t}).$$

In the chaotic dynamical systems,

$$\rho \simeq \mathcal{O}(e^{\lambda t}), \quad (25)$$

with the Lyapunov exponent $\lambda > 0$. Even the last case, where the nearby trajectories separate fast, is quite different from the behavior (24). Indeed, for the exponential separation (25), infinitesimally close trajectories keep shadowing each other and never separate to a finite distance. Definitely, chaos and fully developed turbulence are very different phenomena.

In fact the **explosive separation** (24) has a quite dramatic consequence: it means that, when $Re \rightarrow \infty$, the very concept of a Lagrangian trajectory determined by its initial position in a fixed velocity realization breaks down. This breakdown of the Lagrangian flow is related to a breakdown of the theorem about the existence and unicity of solutions of the ODE (23). The theorem requires $\mathbf{v}(t, \mathbf{x})$ to be Lipschitz in \mathbf{x} , i.e. $|\mathbf{v}(t, \mathbf{x}) - \mathbf{v}(t, \mathbf{y})| \sim |\mathbf{x} - \mathbf{y}|$, whereas at $Re = \infty$ the velocities are only Hölder continuous: $|\mathbf{v}(t, \mathbf{x}) - \mathbf{v}(t, \mathbf{y})| \sim |\mathbf{x} - \mathbf{y}|^\alpha$ with the exponent $\alpha < 1$ ($\alpha \simeq \frac{1}{3}$). We expect that for such velocities one may still maintain a probabilistic description of Lagrangian trajectories with such objects as the probability distribution function (PDF) $P^{t,s}(\mathbf{x}, \mathbf{y}|\mathbf{v})$ of the time s position \mathbf{y} of the trajectory starting at time t at point \mathbf{x} still making sense. $P^{t,s}(\mathbf{x}, \mathbf{y}|\mathbf{v})$ will rather be diffuse, however, than concentrated at one \mathbf{y} . In other words, we expect that, at $Re = \infty$, the Lagrangian trajectories become stochastic already in a fixed realization of the velocity field, providing a mechanism for the energy dissipation in weak solutions of the Euler equation.

This important idea which seems to be a direct consequence in the limit of high Reynolds numbers of the Richardson dispersion law or of the Kolmogorov scaling of velocity differences has been rarely stressed in the past. It has appeared in a study of weak solutions of the Euler equations (in a somewhat different form) and in a recent analytic study of the Lagrangian trajectories in a simple statistical ensemble of velocities with the spatial Hölder continuity of the typical realizations built in. We shall spend the rest of this Lecture by reviewing the latter results with the aim to substantiate the preceding discussion.

Following R. H. Kraichnan who initiated in 1968 the study of transport properties of velocities decorrelated in time, see next Lecture, we shall consider a Gaussian ensemble of velocities with mean zero and 2-point function given by

$$\langle v^\alpha(t, \mathbf{x}) v^\beta(s, \mathbf{y}) \rangle = \delta(t - s) (D_0 \delta^{\alpha\beta} - d^{\alpha\beta}(\mathbf{x} - \mathbf{y})) \quad (26)$$

with D_0 a constant and $d^{\alpha\beta}(\mathbf{x}) \propto r^\xi$ for small $r \equiv |\mathbf{x}|$. $0 < \xi < 2$ will be the parameter of the ensemble. The constant D_0 drops out in the correlations of the velocity differences. E.g.

$$\langle (v^\alpha(t, \mathbf{x}) - v^\alpha(t, \mathbf{0})) (v^\beta(s, \mathbf{x}) - v^\beta(s, \mathbf{0})) \rangle = \delta(t - s) (d^{\alpha\beta}(\mathbf{x}) + d^{\beta\alpha}(\mathbf{x})). \quad (27)$$

One may take

$$D_0 - d^{\alpha\beta}(\mathbf{x}) = \int \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{(\mathbf{k}^2 + L^{-2})^{(d+\xi)/2}} \left(A \frac{k^\alpha k^\beta}{\mathbf{k}^2} + B (\delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{\mathbf{k}^2}) \right) d\mathbf{k} \quad (28)$$

with the infrared cutoff L . For $A = 0$, the typical velocities are incompressible: $\nabla \cdot \mathbf{v} = 0$, whereas for $B = 0$ one obtains potential flows: $\mathbf{v} = \nabla \phi$. It is convenient to characterize the resulting velocity ensemble by, besides ξ , the **compressibility degree**

$$\wp = \frac{\langle (\nabla \cdot \mathbf{v})^2 \rangle}{\langle (\nabla \mathbf{v})^2 \rangle} = \frac{1}{1 + (d-1)\frac{B}{A}}$$

contained between 0 and 1. $\wp = 0$ corresponds to the incompressible case whereas $\wp = 1$ to the potential one.

The above ensemble is not very realistic in its assumption of temporal velocity decorrelation. Recall, however, that the eddy turnover time τ_r is predicted by the Kolmogorov theory to behave as $\sim r^{2/3}$ so that the time correlation of the short scale eddies is small. The Kraichnan ensemble builds in the scaling behavior of the velocities in space with the Hölder continuity $|\mathbf{v}(t, \mathbf{x}) - \mathbf{v}(t, \mathbf{y})| \sim |\mathbf{x} - \mathbf{y}|^{\xi/2}$ of typical realizations, up to logarithmic corrections. To compare to the realistic ensemble of turbulent velocities, note that, for small r , the right hand side of Eq. (27) behaves as

$$\text{const. } \delta\left(\frac{t-s}{r^{2-\xi}}\right) r^{2\xi-2}.$$

We infer then that the velocity v_r of size r eddies scales as $\sim r^{\xi-1}$ corresponding to the eddy turnover time $\tau_r = \frac{r}{v_r} \sim r^{2-\xi}$. Within this limited comparison, $\xi = \frac{4}{3}$ gives the Kolmogorov scaling of velocities. Of course, the Kraichnan ensemble, as Gaussian, is not intermittent. Also its odd-point correlation functions vanish. We shall see in next Lecture, however, that it leads to intermittency in transport phenomena.

In the velocity fields of the Kraichnan ensemble, we shall consider the Lagrangian trajectories, perturbed first by a small noise, i.e. satisfying the stochastic differential equation:

$$\frac{d\mathbf{x}}{ds} = \mathbf{v}(s, \mathbf{x}) + \sqrt{2\kappa} \frac{d\beta(s)}{ds}, \quad (29)$$

where $\beta(s)$ is a d -dimensional Brownian motion and $\kappa > 0$. For a smooth velocity \mathbf{v} , solutions of Eq. (29) form a Markov process which may be characterized by the transition probabilities

$$P^{t,s}(\mathbf{x}, \mathbf{y} | \mathbf{v}) = \overline{\delta(\mathbf{y} - \mathbf{x}_{t,\mathbf{x}}(s))}, \quad (30)$$

where $\mathbf{x}_{t,\mathbf{x}}(s)$ denotes the solution of the differential equation (29) (with s as the running time) passing at time t by \mathbf{x} and the overbar stands for the averaging with respect to the Brownian motion β . The transition probabilities satisfy the linear equation

$$(\partial_t + \mathbf{v}(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} - \kappa \nabla_{\mathbf{x}}^2) P^{t,s}(\mathbf{x}, \mathbf{y} | \mathbf{v}) = 0,$$

with the initial condition $P^{t,t}(\mathbf{x}, \mathbf{y} | \mathbf{v}) = \delta(\mathbf{x} - \mathbf{y})$, so that, in the operator language,

$$P^{t,s}(\mathbf{v}) = \mathcal{T} e^{\int_s^t [-\mathbf{v}(\sigma) \cdot \nabla + \kappa \nabla^2] d\sigma},$$

where \mathcal{T} denotes the time ordering (we have assumed $t \geq s$ here, the case $t < s$ is treated similarly).

As noticed by Y. Le Jan and O. Raimond, the transition probabilities $P^{t,s}(\mathbf{v})$ still make sense for velocities \mathbf{v} of the Kraichnan ensemble, despite the poor regularity properties of the latter. These authors have rewritten the above expression for $P^{t,s}(\mathbf{v})$ in the Wick-ordered form:

$$\begin{aligned}
P^{t,s}(\mathbf{v}) &= : \mathcal{T} e^{\int_s^t [-\mathbf{v}(\sigma) \cdot \nabla + (\kappa + \frac{1}{2} D_0) \nabla^2] d\sigma} : \\
&= \sum_{n=0}^{\infty} (-1)^n \int_s^t d\sigma_n \dots \int_s^{\sigma_2} d\sigma_1 : e^{(\kappa + \frac{1}{2} D_0)(t - \sigma_n) \nabla^2} \mathbf{v}(\sigma_n) \cdot \nabla e^{(\kappa + \frac{1}{2} D_0)(\sigma_n - \sigma_{n-1}) \nabla^2} \dots \\
&\quad \dots e^{(\kappa + \frac{1}{2} D_0)(\sigma_2 - \sigma_1) \nabla^2} \mathbf{v}(\sigma_1) \cdot \nabla e^{(\kappa + \frac{1}{2} D_0)(\sigma_1 - s) \nabla^2} : , \tag{31}
\end{aligned}$$

which is an infinite sum of Wick ordered monomials of \mathbf{v} that may be represented by the diagrams of Fig. 2.

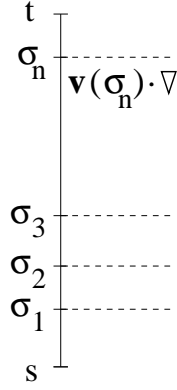


Fig. 2

The homogeneous Wick polynomials of different degrees are orthogonal in the L^2 scalar product w.r.t. the Gaussian measure of the velocity process. By establishing the bound

$$\langle |P^{t,s}(\mathbf{v})g|^2 \rangle \leq e^{\frac{1}{2} D_0(t-s) \nabla^2} |g|^2$$

where g are functions on \mathbf{R}^d , one shows then that the series giving $P^{s,t}(\mathbf{v})f$ converges in the space of square-integrable functionals of \mathbf{v} , so also for almost all (a.a.) \mathbf{v} , as long as g is bounded. It defines for a.a. velocities the transition probabilities $P^{t,s}(\mathbf{x}, \mathbf{y} | \mathbf{v})$ of a Markov process which are continuous as functions of $\kappa \geq 0$. Note that

$$P_1^{t,s}(\mathbf{x}, \mathbf{y}) \equiv \langle P^{t,s}(\mathbf{v})(\mathbf{x}, \mathbf{y}) \rangle = e^{(\kappa + \frac{1}{2} D_0)|t-s| \nabla^2}(\mathbf{x}, \mathbf{y}). \tag{32}$$

The essential question that we want to address is about the nature of the Markov process obtained in the limit $\kappa \rightarrow 0$. Are the limiting transition probabilities concentrated at single points \mathbf{y} leading to deterministic Lagrangian trajectories determined, in a fixed velocity realization, by the initial position or, on the contrary, do they stay diffuse? A way to study this question is to examine the joint PDF (probability distribution function) of the equal-time values of two solutions of Eq. (29) averaged over the velocity ensemble:

$$P_2^{t,s}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_2) = \langle P^{t,s}(\mathbf{x}_1, \mathbf{y}_1 | \mathbf{v}) P^{t,s}(\mathbf{x}_2, \mathbf{y}_2 | \mathbf{v}) \rangle.$$

The average on the right hand side is given by the sum of terms described by the diagrams of Fig. 3 with the broken-line propagators corresponding to the spatial part ($D_0 - d(\cdot)$) of the velocity 2-point functions (26). The whole sum becomes the perturbative expansion for the heat kernel of the second order differential operator:

$$P_2^{t,s}(\underline{\mathbf{x}}; \underline{\mathbf{y}}) = e^{-|t-s|M_2}(\underline{\mathbf{x}}; \underline{\mathbf{y}}), \quad (33)$$

where

$$\begin{aligned} M_2 &= -(\kappa + \frac{1}{2}D_0)(\nabla_{\mathbf{x}_1}^2 + \nabla_{\mathbf{x}_2}^2) - (D_0\delta^{\alpha\beta} - d^{\alpha\beta}(\mathbf{x}_1 - \mathbf{x}_2))\nabla_{x_1^\alpha}\nabla_{x_2^\beta} \\ &= d^{\alpha\beta}(\mathbf{x}_1 - \mathbf{x}_2)\nabla_{x_1^\alpha}\nabla_{x_2^\beta} - \kappa(\nabla_{\mathbf{x}_1}^2 + \nabla_{\mathbf{x}_2}^2) + \frac{1}{2}D_0(\nabla_{\mathbf{x}_1} + \nabla_{\mathbf{x}_2})^2. \end{aligned} \quad (34)$$

The last term drops out in the translation-invariant sector. In other words, two solutions of Eq. (29) undergo, in their relative motion, an effective diffusion with the diffusion coefficient dependent on their relative position.

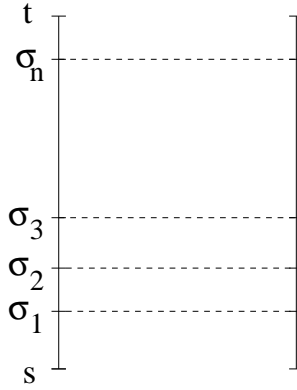


Fig. 3

The PDF $P_2^{t,s}(r; \rho)$ of the distance ρ between the time s positions of two solutions, given their time t distance r , is expressed by the heat kernel of the operator M_2 restricted to the translation and rotation invariant sector. The latter becomes an explicit second order differential operator in the radial variable:

$$M_2^{inv} = -C r^{\xi-a} \partial_r r^a \partial_r - 2\kappa r^{-d+1} \partial_r r^{d-1} \partial_r \quad (35)$$

with the exponent $a = a(\xi, \varphi)$ (for simplicity, we give the formula after the removal of the infrared cutoff L in Eq. (28)). The operator M_2^{inv} may be transformed by a change of variables and a similarity transformation to a Schrödinger operator on the half-line. In particular,

$$\lim_{\kappa \rightarrow 0} M_2^{inv} = C' u^c \left[-\partial_u^2 + \frac{b^2 - \frac{1}{4}}{u^2} \right] u^{-c}, \quad (36)$$

where $u = r^{\frac{2-\xi}{2}}$ and $b = b(\xi, \varphi)$, $c = c(\xi, \varphi)$. The limit $\kappa \rightarrow 0$ of the PDF $P_2^{t,s}(r, \rho)$ can be explicitly controlled. Two different regimes appear in this limit.

1. Weakly compressible regime

For weak compressibility $\wp < \frac{d}{\xi^2}$, which corresponds to $b < 1$, the distance PDF $P_2^{t,s}(r, \rho)$ is an integrable function of ρ and stays such in the limit $r \rightarrow 0$:

$$\lim_{r \rightarrow 0} \lim_{\kappa \rightarrow 0} P_2^{t,s}(r; \rho) d\rho \propto \left(\frac{\rho^{2-\xi}}{|t-s|}\right)^{1-b} e^{-\frac{\rho^{2-\xi}}{4C'|t-s|}} \frac{d\rho}{\rho}. \quad (37)$$

This behavior excludes concentration of the transition probabilities $P^{t,s}(\mathbf{x}, \mathbf{y}|\mathbf{v})$ at single points \mathbf{y} . In particular, we obtain the Richardson dispersion law in the form

$$\lim_{r \rightarrow 0} \lim_{\kappa \rightarrow 0} \int \rho^2 P_2^{0,t}(r, \rho) d\rho \propto t^{\frac{2}{2-\xi}}$$

indicating an explosive separation of the Lagrangian trajectories and reproducing, for $\xi = \frac{4}{3}$, the mean growth (24). As we see, the trajectories in a fixed typical realization of the velocity fields are not determined by the initial position but rather form a Markov process with diffuse transition probabilities $P^{t,s}(\mathbf{x}, \mathbf{y}|\mathbf{v})$, as predicted above, see Fig. 4.

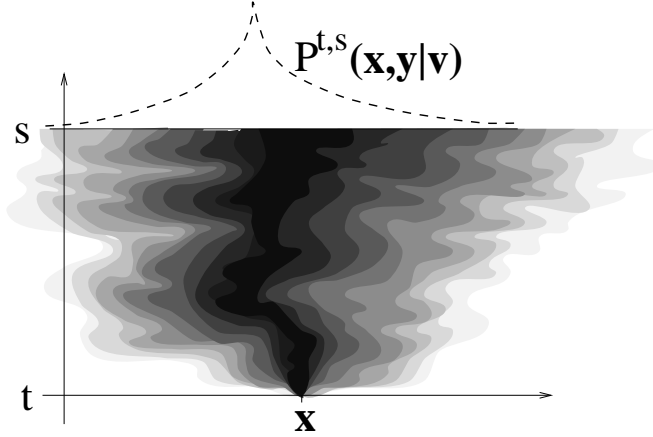


Fig. 4

2. Strongly compressible regime

For strong compressibility $\wp \geq \frac{d}{\xi^2}$, i.e. for $b \geq 1$, one observes a different behavior due to the strong repulsive singularity at $u = 0$ in the operator (36):

$$\lim_{\kappa \rightarrow 0} P_2^{t,s}(r; \rho) d\rho = p^{t,s}(r) \delta(\rho) d\rho + \text{regular}$$

with the coefficient $p^{t,s}(r)$ of the delta-function converging to 1 when $r \rightarrow 0$ or $|t - s| \rightarrow \infty$. In particular,

$$\lim_{r \rightarrow 0} \lim_{\kappa \rightarrow 0} P_2^{t,s}(r; \rho) d\rho = \delta(\rho) d\rho$$

in this regime implying the concentration of the transition probabilities $P^{t,s}(\mathbf{x}, \mathbf{y}|\mathbf{v})$ at single points \mathbf{y} and the existence, in a fixed typical velocity realization, of Lagrangian trajectories determined by their initial positions. The presence of the singular term in $P_2^{t,s}(r; \rho)$ for $\kappa = 0$ indicates, however, an **implosive collapse** of distinct Lagrangian trajectories with a positive probability which grows in time, see Fig. 5.

We infer that in Hölder-continuous velocity fields, there is a competition of the tendency of two Lagrangian trajectories to separate or to collapse explosively. For weak compressibility, this is the first tendency that wins. The trapping effects increase, however, with the degree of compressibility and lead, in the Kraichnan ensemble, to a sharp transition in the behavior of the trajectories at $\varphi = \frac{d}{\xi^2}$.

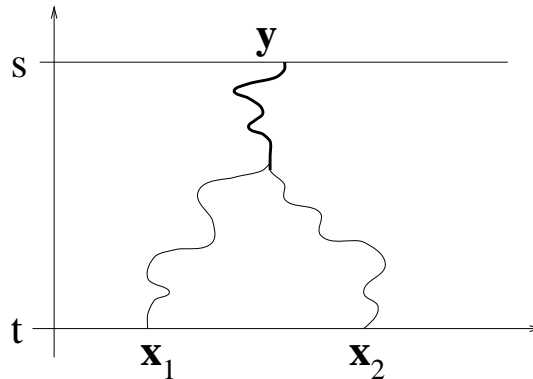


Fig. 5

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LECTURE 4

What are the consequences on the hydrodynamical properties of the flows of the observed dramatic behaviors of Lagrangian trajectories, violating the Newton-Leibniz paradigm about existence and uniqueness of solutions of the ODE's? The answer to this question may be the clue to a consistent theory of developed turbulence. Here, we shall content ourselves with discussing the transport properties in the flows induced by the velocities of the Kraichnan ensemble. We shall see that these properties differ drastically for weak and strong compressibility as a result of different behaviors of the Lagrangian trajectories. For concreteness, we shall look at the passive transport of the scalar quantity $\theta(t, \mathbf{x})$ (called tracer) whose evolution is described by the advection-diffusion equation

$$\partial_t \theta + (\mathbf{v} \cdot \nabla) \theta - \kappa \nabla^2 \theta = f, \quad (38)$$

where $f(t, \mathbf{x})$ denotes now a scalar source. In the incompressible case, θ may also describe the temperature field or the density of pollutant. The passivity of the advection means that the back reaction of θ on the velocity is ignored.

When the velocity field is sufficiently smooth, it is easy to solve the above linear equation for θ . For $\kappa = 0$ and $f = 0$, the scalar is simply carried by the Lagrangian flow

$$\theta(t, \mathbf{x}) = \theta(s, \mathbf{x}_{t,\mathbf{x}}(s)),$$

where $\mathbf{x}_{t,\mathbf{x}}(s)$ is the Lagrangian trajectory passing at time t by point \mathbf{x} . Note that the forward evolution of θ corresponds to the backward Lagrangian flow. In the presence of the source f , the scalar is also created or depleted along the trajectory:

$$\theta(t, \mathbf{x}) = \theta(s, \mathbf{x}_{t,\mathbf{x}}(s)) + \int_s^t f(\sigma, \mathbf{x}_{t,\mathbf{x}}(\sigma)) d\sigma.$$

Finally, when $\kappa \neq 0$, $\mathbf{x}_{t,\mathbf{x}}(s)$ should be taken as a solution of the equation (29) for the Lagrangian trajectories perturbed by the Brownian motion and the above formulae should be averaged over the latter. Thus

$$\begin{aligned} \theta(t, \mathbf{x}) &= \int \overline{\delta(\mathbf{y} - \mathbf{x}_{t,\mathbf{x}}(s))} \theta(s, \mathbf{y}) d\mathbf{y} + \int_s^t \left(\int \overline{\delta(\mathbf{y} - \mathbf{x}_{t,\mathbf{x}}(\sigma))} f(\sigma, \mathbf{y}) d\mathbf{y} \right) d\sigma \\ &= \int P^{t,s}(\mathbf{x}, \mathbf{y}|\mathbf{v}) \theta(s, \mathbf{y}) d\mathbf{y} + \int_s^t \left(\int P^{t,\sigma}(\mathbf{x}, \mathbf{y}|\mathbf{v}) f(\sigma, \mathbf{y}) d\mathbf{y} \right) d\sigma, \end{aligned} \quad (39)$$

see Eq. (30). The right hand side still makes sense for a.a. velocities of the Kraichnan ensemble and defines a weak solution of the linear differential equation (38), i.e. the one satisfying the equation in the distributional sense⁴.

⁴Eq. (38) is an infinite-dimensional stochastic differential equation due to the white temporal dependence of \mathbf{v} and it should be treated according to the Stratonovich prescription; the weakness of the solution is, however, due to its poor spatial regularity resulting from the non-differentiability in space of typical velocities and the related stochastic character of the Lagrangian trajectories

First, let us assume that we are given a random distribution of the scalar at time s , independent of the (later) velocities, and we wish to study its distribution at the later time t . In free decay, i.e. in the absence of the source f , by taking averages over the initial distribution and over the velocities, we obtain for the 1-point function of θ :

$$\langle \theta(t, \mathbf{x}) \rangle = \int P_1^{t,s}(\mathbf{x}, \mathbf{y}) \langle \theta(s, \mathbf{y}) \rangle d\mathbf{y} = \int e^{-(s-t)(\kappa + \frac{1}{2}D_0)\nabla^2}(\mathbf{x}, \mathbf{y}) \langle \theta(s, \mathbf{y}) \rangle d\mathbf{y},$$

see Eq. (32). In other words, the 1-point function decays diffusively. Note that the initial diffusivity κ is increased by the **eddy diffusivity** $\frac{1}{2}D_0$. Similarly, for the 2-point function,

$$\langle \theta(t, \mathbf{x}_1) \theta(t, \mathbf{x}_2) \rangle = \int P_2^{t,s}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_2) \langle \theta(s, \mathbf{y}_1) \theta(s, \mathbf{y}_2) \rangle d\mathbf{y}_1 d\mathbf{y}_2 \quad (40)$$

and, for the N -point one,

$$\langle \prod_{n=1}^N \theta(t, \mathbf{x}_n) \rangle = \int P_N^{t,s}(\underline{\mathbf{x}}; \underline{\mathbf{y}}) \langle \prod_{n=1}^N \theta(s, \mathbf{y}_n) \rangle d\underline{\mathbf{y}},$$

where

$$P_N^{t,s}(\underline{\mathbf{x}}; \underline{\mathbf{y}}) = \langle \prod_{n=1}^N P^{t,s}(\mathbf{x}_n, \mathbf{y}_n | \mathbf{v}) \rangle = P_N^{s,t}(\underline{\mathbf{x}}; \underline{\mathbf{y}})$$

is the joint PDF of the time s positions $\underline{\mathbf{y}}$ of the Lagrangian trajectories, given their time t positions $\underline{\mathbf{x}}$, see Fig. 6 (the last equality follows from the stationarity and time-reflection invariance of the Kraichnan ensemble).

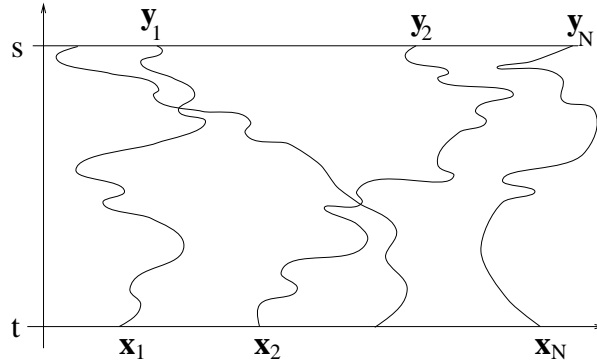


Fig. 6

It is easy to see that the PDF's $P_N^{t,s}(\underline{\mathbf{x}}; \underline{\mathbf{y}})$ are again given by the heat kernels of second order differential operators:

$$P_N^{t,s}(\underline{\mathbf{x}}; \underline{\mathbf{y}}) = e^{-|t-s|M_N}(\underline{\mathbf{x}}; \underline{\mathbf{y}}),$$

where

$$M_N = \sum_{n < m} d^{\alpha\beta}(\mathbf{x}_n - \mathbf{x}_m) \nabla_{x_n^\alpha} \nabla_{x_m^\beta} - \kappa \sum_{n=1}^N \nabla_{\mathbf{x}_n}^2 + \frac{1}{2} D_0 \left(\sum_n \nabla_{\mathbf{x}_n} \right)^2.$$

Assuming a homogeneous and isotropic initial distribution of θ , we may replace the PDF $P_2^{t,s}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_1, \mathbf{y}_2)$ in Eq. (40) by its translation- and rotation-invariant version $P_2^{t,s}(r, \rho)$. In particular, taking the limit $r \rightarrow 0$, we infer that, for $\kappa = 0$ and in the weakly compressible case $\wp < \frac{d}{\xi^2}$, the mean scalar energy density

$$\bar{e}_\theta(t) \equiv \langle \frac{1}{2} \theta(t)^2 \rangle = \frac{1}{2} \int P_2^{t,s}(0, \rho) \langle \theta(s, \mathbf{y}) \theta(s, \mathbf{0}) \rangle d\rho < \bar{e}_\theta(s) \quad (41)$$

if the 2-point function $\langle \theta(s, \mathbf{y}) \theta(s, \mathbf{0}) \rangle$ decays for large $\rho = |\mathbf{y}|$. Indeed, the latter is bounded by its value at $\rho = 0$ so that the relation (41) follows since $P_2^{t,s}(0, \rho)$ is a strictly positive probability density, see Eq. (37). Hence the mean energy density of the scalar decreases with time (i.e. is dissipated) even for $\kappa = 0$. On the contrary, in the strongly compressible case $\wp \geq \frac{d}{\xi^2}$, the limit PDF $P_2^{t,s}(0, \rho) = \delta(\rho)$ and the mean energy density is conserved in the unforced evolution at $\kappa = 0$:

$$\bar{e}_\theta(t) = \bar{e}_\theta(s).$$

In the presence of the source f which, as before, we shall assume random Gaussian, independent of velocities and the initial distribution of θ , with mean zero and covariance

$$\langle f(t, \mathbf{x}) f(s, \mathbf{y}) \rangle = \delta(t - s) \chi\left(\frac{|\mathbf{x} - \mathbf{y}|}{L}\right), \quad (42)$$

the 1-point function of the scalar diffuses as before and for the 2-point function, we obtain from Eq. (39)

$$\langle \theta(t, \mathbf{x}_1) \theta(t, \mathbf{x}_2) \rangle = \int P_2^{t,s}(\underline{\mathbf{x}}; \underline{\mathbf{y}}) \langle \theta(s, \mathbf{y}_1) \theta(s, \mathbf{y}_2) \rangle d\underline{\mathbf{y}} + \int_s^t \left(\int P_2^{t,\sigma}(\underline{\mathbf{x}}; \underline{\mathbf{y}}) \chi\left(\frac{|\mathbf{y}_1 - \mathbf{y}_2|}{L}\right) d\underline{\mathbf{y}} \right) d\sigma. \quad (43)$$

The 2-point function solves now the equation

$$\begin{aligned} \partial_t \langle \theta(\mathbf{x}_1) \theta(\mathbf{x}_2) \rangle &= -M_2 \langle \theta(\mathbf{x}_1) \theta(\mathbf{x}_2) \rangle + \chi\left(\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{L}\right) \\ &= -d^{\alpha\beta}(\mathbf{x}_1 - \mathbf{x}_2) \langle \nabla_\alpha \theta(\mathbf{x}_1) \nabla_\beta \theta(\mathbf{x}_2) \rangle - 2\kappa \langle \nabla \theta(\mathbf{x}_1) \cdot \nabla \theta(\mathbf{x}_2) \rangle + \chi\left(\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{L}\right), \end{aligned} \quad (44)$$

see Eqs. (33) and (34). This is the passive scalar counterpart of the NS relation (8) that we discussed in Lecture 2. In the weakly compressible regime $\wp < \frac{d}{\xi^2}$, the scalar 2-point function reaches a stationary state. Arguing as before for the NS case, we infer in this state the energy balance

$$\bar{e}_\theta = \frac{1}{2} \chi(0)$$

for the scalar mean dissipation rate $\bar{e}_\theta = \langle \kappa (\nabla \theta)^2 \rangle$ and the relation

$$\frac{1}{2} d^{\alpha\beta}(\mathbf{x}_1 - \mathbf{x}_2) \nabla_{x_1^\alpha} \nabla_{x_2^\beta} \langle \theta(\mathbf{x}_1) \theta(\mathbf{x}_2) \rangle = \frac{1}{2} \chi\left(\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{L}\right).$$

The latter may be easily solved for the stationary 2-point function of the scalar giving

$$\langle \theta(\mathbf{x}) \theta(\mathbf{0}) \rangle = A_2(\chi) L^{2-\xi} - \text{const.} \bar{e}_\theta r^{2-\xi} + \mathcal{O}(L^{-2}), \quad (45)$$

where $r \equiv |\mathbf{x}|$ or, for the scalar 2-point structure function,

$$S_2(r) \equiv \langle (\theta(\mathbf{x}) - \theta(\mathbf{0}))^2 \rangle \propto \bar{e}_\theta r^{2-\xi} \quad (46)$$

for $\kappa = 0$ and $r \ll L$. This is an analogue of the Kolmogorov $\frac{4}{5}$ law. It may be strengthened to the operator product expansion for the $\kappa \rightarrow 0$ limit of the dissipation operator $\epsilon_\theta = \kappa(\nabla\theta)^2$

$$\epsilon_\theta(\mathbf{x}) = \lim_{\mathbf{y} \rightarrow \mathbf{x}} d^{\alpha\beta}(\mathbf{x} - \mathbf{y}) \nabla_\alpha \theta(\mathbf{x}) \nabla_\beta \theta(\mathbf{y}) \Big|_{\kappa=0} \quad (47)$$

valid inside expectations in the limit $\kappa \rightarrow 0$ and also in fixed realizations of θ . Eq. (47) expresses the dissipative anomaly in the weakly compressible regime of the Kraichnan model, analogous to the dissipative anomaly (12) for the Navier-Stokes case. One may also check directly the approximate constancy of the scalar energy flux towards large wavenumbers, establishing the existence of a direct scalar energy cascade.

In the strongly compressible regime $\wp > \frac{d}{\xi^2}$, the behavior of the scalar 2-point function (43) in the limit $\kappa \rightarrow 0$ is quite different. Now the 2-point function does not stabilize but has a constant contribution growing linearly in time. The dissipation rate vanishes and scalar energy is pumped into the constant mode at a constant rate equal to the injection rate $\frac{1}{2}\chi(0)$. This signals the presence of an inverse cascade of scalar energy towards small wavenumbers. The 2-point structure function of the scalar stabilizes, however, and its stationary limit satisfies the equation

$$M_2 \langle (\theta(\mathbf{x}) - \theta(\mathbf{0}))^2 \rangle = 2(\chi(0) - \chi(\frac{|\mathbf{x}|}{L}))$$

from which one infers that

$$S_2(r) \propto r^{2-\xi}$$

for $r \gg L$.

As for intermittency of the scalar statistics, the two regimes also show very different behaviors. The question here is whether the higher structure functions of the scalar $S_N(r) \equiv \langle (\theta(\mathbf{x}) - \theta(0))^N \rangle$ scale with powers $\frac{N}{2}(2 - \xi)$ as the dimensional analysis would suggest, in analogy to the Kolmogorov theory. Although, by assumption, in the Kraichnan model there is no intermittency in the statistics of the velocity differences, the numerical studies of the incompressible model indicate strong intermittency of the scalar differences signaled by anomalous values of the scaling exponents. Unlike in the NS case, we have now some analytic understanding of this phenomenon.

First, it is not difficult to see that the higher equal-time correlators of the scalar satisfy the evolution equations generalizing Eq. (44):

$$\partial_t \langle \prod_{n=1}^N \theta(\mathbf{x}_n) \rangle = -M_N \langle \prod_{n=1}^N \theta(\mathbf{x}_n) \rangle + \sum_{p < q} \langle \prod_{n \neq p, q} \theta(\mathbf{x}_n) \rangle \chi(\frac{|\mathbf{x}_p - \mathbf{x}_q|}{L}). \quad (48)$$

In the weakly compressible regime, the correlation functions stabilize for long time and, besides, we expect that the limits $t \rightarrow \infty$ and $\kappa \rightarrow 0$ commute. The stationary equal time correlations satisfy a similar equation but with the vanishing left hand side. By inverting operators M_N , we may then compute the stationary N -point functions of the scalar recursively, a rare situation, indeed, since in most hydrodynamical problems the evolution equations for the correlation functions, called the Hopf equations, do not close. A semi-rigorous analysis shows that, for small ξ in the limit $\kappa \rightarrow 0$ and on short distances or for large injection scale L ,

$$\langle \prod_{n=1}^N \theta(\mathbf{x}_n) \rangle = A_N(\chi) L^{\frac{N}{2}(2-\xi)-\zeta_N} \varphi_N(\underline{\mathbf{x}}) + \mathcal{O}(L^{-2+\mathcal{O}(\xi)}) + \dots,$$

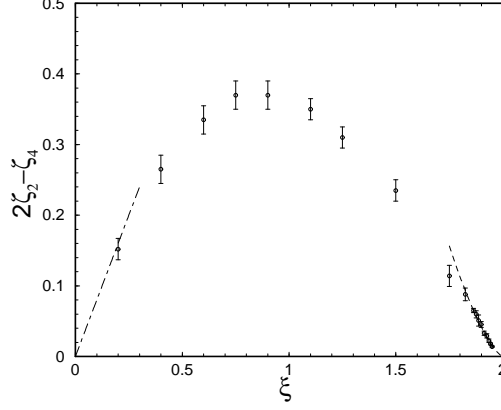


Fig. 7

where $\varphi_N(\underline{\mathbf{x}})$ are scaling zero modes of the operators M_N , i.e.

$$M_N \varphi_N(\underline{\mathbf{x}}) = 0, \quad \varphi_N(\lambda \underline{\mathbf{x}}) = \lambda^{\zeta_N} \varphi_N(\underline{\mathbf{x}})$$

with

$$\zeta_N = \frac{N}{2}(2 - \xi) - \frac{N(N-2)(1+2\varphi)}{2(d+2)}\xi + \mathcal{O}(\xi^2).$$

The coefficients $A_N(\chi)$ are non-universal amplitudes and the dots denote terms that do not depend of, at least, one variable and, as such, do not contribute to the correlations of scalar differences. In particular, the structure functions

$$S_N(r) \propto L^{\frac{N}{2}(2-\xi)-\zeta_N} r^{\zeta_N}.$$

This **zero mode dominance** of the stationary higher-point functions of the scalar (note that such zero modes drop out in the stationary version of Eq. (48)) has been exhibited by the perturbative analysis of the Green functions of operators M_N around $\xi = 0$. It has been confirmed by perturbative analyses in powers of the inverse dimension and of $(2 - \xi)$ and by numerical results, see Fig. 7 representing the values of the simulations by Frisch-Mazzino-Vergassola of the 4-point function anomalous exponent in the three-dimensional incompressible Kraichnan model.

What is the physical meaning of the zero modes of the operators M_N that dominate the short-distance asymptotics of the N -point functions of scalar differences? They are **slow modes** of the effective diffusion of Lagrangian trajectories with generators M_N . Indeed, for generic scaling function $\psi_N(\underline{\mathbf{x}})$ of scaling dimension σ_N , viewed as a function of time t positions of the Lagrangian trajectories, the effective time evolution is

$$\langle \psi_N \rangle_t \equiv \int P_N^{0,t}(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \psi_N(\underline{\mathbf{y}}) d\mathbf{y} \sim t^{\frac{\sigma_N}{2-\xi}} \quad (49)$$

for large t , i.e. it exhibits a **super-diffusive** growth. But if $\psi_N = \phi_N$ is a zero mode of M_N then the above expectation is conserved in time (such statistically conserved modes are accompanied by descendent ones for which the time growth is slower than (49)).

In the strongly compressible phase with the inverse cascade of scalar energy, the behavior of the higher structure functions is different. In fact, only the lower ones stabilize, but the ones that do, scale normally on large distances. In this regime one can find exactly the stationary form of the PDF of the scalar difference:

$$\langle \delta(\vartheta - \frac{\theta(\mathbf{x}) - \theta(\mathbf{0})}{r^{(2-\xi)/2}}) \rangle \propto [\chi(0) + C' \vartheta^2]^{-b-\frac{1}{2}}$$

for $r \gg L$. Its scaling form indicates that there is no intermittency in the inverse cascade of the scalar (the non-Gaussianity is scale-independent). Its poor decay at infinity corresponds to the fact that only lower structure functions reach a stationary regime.

As we see, the transport of a scalar tracer by velocities distributed according to the Kraichnan ensemble shows two different phases characterized by different direction of the scalar energy cascades and different degrees of intermittency. The phase transition occurs at the value $\wp = \frac{d}{\xi^2}$ of the compressibility degree, where the behavior of the Lagrangian trajectories changes drastically from the explosive separation to the implosive collapse. These two phases are quite reminiscent of the behavior of the three dimensional versus two-dimensional developed turbulence. That suggests that one should put more stress on the Lagrangian methods in studying the latter, especially on the properties of the Lagrangian flow in the weak solutions of the Euler equation. Of course the NS or the Euler equation, unlike the scalar advection one, are non-linear. They describe velocity fields that are not only carried by their own Lagrangian trajectories but also stretched and there are non-local effects due to pressure. Some of these effects, however, may be studied already in various models of passive advectons (passive vectors, linearized NS equation, etc.). It seems that the study of easy models of turbulence has a potential to teach us important lessons that we have to master to stand a chance of solving the fully-fledged problem of developed turbulence.

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